Dissipation and Topologically Massive Gauge Theories in Pseudoeuclidean Plane

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Abstract

In the pseudo-euclidean metrics Chern-Simons gauge theory in the infrared region is found to be associated with dissipative dynamics. In the infrared limit the Lagrangian of 2+1 dimensional pseudo-euclidean topologically massive electrodynamics has indeed the same form of the Lagrangian of the damped harmonic oscillator. On the hyperbolic plane a set of two damped harmonic oscillators, each other time-reversed, is shown to be equivalent to a single undamped harmonic oscillator. The equations for the damped oscillators are proven to be the same as the ones for the Lorentz force acting on two particles carrying opposite charge in a constant magnetic field and in the electric harmonic potential. This provides an immediate link with Chern-Simons-like dynamics of Bloch electrons in solids propagating along the lattice plane with hyperbolic energy surface. The symplectic structure of the reduced theory is finally discussed in the Dirac constrained canonical formalism.

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1. Introduction

It is well known that quantum gauge field theory in 2+1 dimensional space-time admits nontrivial topological structure defined by the Chern-Simons term[1,2,3] and, in the euclidean metrics, it provides an useful theoretical framework in the description of interesting phenomena in planar condensed matter physics as, for example, fractional statistics particles[4], the quantum Hall effect, high T_c superconductivity[5].

The topological properties of the theory are connected with the form of Chern-Simons term, which is independent of the space-time metrics. The metric structure, however, is crucial in determinig other physical features which, as we will see, are related with the canonical structure of the theory.

In ref.[2] topological Chern-Simons theory has been studied by resorting to the quantum mechanical analysis of an oscillator model. The purpose of this paper is to show that in pseudo-euclidean space the Lagrangian density of 2+1 dimensional gauge theory with Chern-Simons term (topological mass term) and Proca mass term can be interpreted in the infrared region[6] as the Lagrangian density of the damped harmonic oscillator, provided the topological Chern-Simons mass parameter is identified with the damping parameter: In the pseudo-euclidean metrics the Chern-Simons gauge theories can be therefore associated with dissipative dynamics.

Such a conclusion is interesting in several respects.

From one side indeed, dissipative systems are of great interest in high energy physics (in quark-gluons physics, for example), in the early universe physics as well as in many body theories, in phase transition phenomena, in quantum optics and in many practical applications of quantum field theory at non-zero temperature. The possibility, then, to exhibit the gauge theory structure and topologically nontrivial features underlying dissipative phenomena is very appealing. On the other side, realizing vice versa that topologically massive electrodynamics in pseudo-euclidean metrics can be interpreted as a dissipative dynamics can be as well interesting since new perspectives open up for Chern-Simons gauge theories with consequent much wider range of possible applications.

In recent years the problem of the canonical quantization of the damped harmonic oscillator in the operator formalism has been intensively studied and some further light has been shed on quantum dissipative phenomena[7-9]. By resorting to the results of Bateman on the canonical formalism for par-

tial differential equations with dissipative terms[10], of Umezawa, Takahashi and collaborators on finite temperature quantum field theory in the operator formalism[11] and of Feshbach and Tikochinski on the quantum mechanics of the damped harmonic oscillator[12], it has been shown that the canonical quantization of the damped oscillator can be properly performed by doubling the phase-space degrees of freedom in the framework of quantum field theory(QFT) [7]. The space of the states has been shown to split into unitarily inequivalent representations of the canonical commutation relations, and the non-unitary character of the irreversible time evolution is expressed as tunneling among such inequivalent representations. The vacuum has been found to have the structure of SU(1,1) time-dependent coherent state and its statistical and thermodynamical properties have been recognized[7] to be the ones of the thermal vacuum in Thermo Field Dynamics[11].

The SU(1,1) group structure underlying the theory signals the rôle played by the pseudo-euclidean metrics in the canonical formalism for quantum dissipation. In Sec.2 we indeed show the strict relationship between the doubling of the degrees of freedom for dissipative systems and the pseudo-euclidean metrics. In particular we prove that the system made of the damped harmonic oscillator and of its time-reversed image (the *doubled* damped oscillator) globally behaves as a closed system described by a single *undamped* harmonic oscillator. Vice versa, by use of hyperbolic coordinates (pseudo-euclidean metrics) an harmonic oscillator can be "decomposed" into two damped harmonic oscillators (each other time-reversed). Moreover, we also show that by convenient representation of the dissipative factor by a magnetic field[13] and by choosing pseudo-euclidean metrics the equations for the damped harmonic oscillator and of its time-reversed image can be cast in the form of the Lorentz force equations for particles of opposite electric charge $e_1 = -e_2 = e$, respectively. In Sec.3 we show that, by considering its infrared properties, the Lagrangian for the topologically massive electrodynamics reproduces the Lagrangian for the damped harmonic oscillator with doubled degrees of freedom when the pseudo-euclidean metrics is used. As an explicit application we study in Sec.4 the dynamics of the Bloch electrons in solids propagating along the plane of the lattice with hyperbolic energy surface[14]. In this example the Lorentz force (the Chern-Simons term) clearly plays the rôle of the damping term and again the system Lagrangian density is recognized to be the one of the damped harmonic oscillator. In Sec.5 we analyze the canonical structure in the context of constrained canonical formalism of

Dirac. Finally, Sec.6 is devoted to conclusive remarks.

2. The damped harmonic oscillator and the pseudo-euclidean metrics

In order to introduce our subsequent discussion on the Chern-Simons term, we need to show how the pseudo-euclidean metrics arises in the canonical formalism for dissipative systems. We consider the simple case of one-dimensional damped harmonic oscillator (dho):

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \quad , \tag{2.1a}$$

with time independent m, γ and k.

It is well known[7-12] that to set up the canonical formalism for dissipative systems the doubling of the degrees of freedom is required in such a way to complement the given dissipative system with its time-reversed image, thus obtaining a globally closed system for which the Lagrangian formalism is well defined. The time-reversed image of the given system plays the rôle of reservoir or thermal bath into which the energy dissipated by the original system flows. In the case of system (2.1a) the doubling of the x degree of freedom leads to consider the dho in the *doubled* y coordinate

$$m\ddot{y} - \gamma\dot{y} + ky = 0 \quad . \tag{2.1b}$$

The system of damped harmonic oscillators (2.1a) and its *time-reversed* $(\gamma \to -\gamma)$ image (2.1b) is then a closed system described by the Lagrangian density

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy \quad . \tag{2.2}$$

The canonically conjugate momenta p_x and p_y can now be introduced as customary in the Lagrangian formalism:

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}y \qquad , \qquad p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}x \quad , \tag{2.3}$$

and the dynamical variables $\{x, p_x; y, p_y\}$ span the new phase-space. The Hamiltonian is then

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{1}{m}p_x p_y + \frac{1}{2m}\gamma\left(yp_y - xp_x\right) + \left(k - \frac{\gamma^2}{4m}\right)xy \quad . \tag{2.4}$$

The common frequency of the two oscillators (2.1a) and (2.1b) will be denoted by $\Omega \equiv \left[\frac{1}{m}\left(k - \frac{\gamma^2}{4m}\right)\right]^{\frac{1}{2}}$, and is assumed to be real, i.e. $k > \frac{\gamma^2}{4m}$, corresponding to the case of *no* overdamping.

Canonical quantization may then be performed by introducing the commutators $[\,x, p_x\,] = i\,\hbar = [\,y, p_y\,] \quad , \quad [\,x, y\,] = 0 = [\,p_x, p_y\,]$, and the corresponding sets of annihilation and creation operators

$$a \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right) \quad ; \quad a^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right) ;$$

$$b \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right) \quad ; \quad b^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}}\left(\frac{p_y}{\sqrt{m}} + i\sqrt{m}\Omega y\right) , \tag{2.5}$$

with

$$[\,a, a^\dagger\,] = 1 = [\,b, b^\dagger\,] \quad , \quad [\,a, b\,] = 0 = [\,a, b^\dagger\,] \quad . \tag{2.6}$$

A detailed account of the quantization procedure in QFT may be found in refs. [7,8].

We stress that time-reversal of y-oscillator with respect to x-oscillator implies that the y-oscillator acts indeed as the sink where the energy dissipated by the x-oscillator flows and thus as the reservoir or heat bath coupled to the x-oscillator[1]. This means that in order to set up the canonical formalism for the damped oscillator (2.1a) the details of the reservoir dynamics need not to be specified; the only requirement to be met is that the reservoir must receive *all* the energy flux outgoing from the x-system so that the coupled $(x-y)$-system globally behaves as a closed (non dissipative) system: In other words, we can show that the set of eqs.(2.1) is formally equivalent to the equation for the (*undamped*) one dimensional harmonic oscillator, say $r(t)$, representing the global $(x-y)$-system:

$$m\ddot{r} + Kr = 0 \quad , \tag{2.7}$$

with

$$K \equiv m\Omega^2 \quad . \tag{2.8}$$

[1] The linear system with "negative damping" could be realized physically, e.g., as the Froude pendulum or the electronic feedback generator [25].

Such an equivalence is easily proven if one uses hyperbolic coordinates $x_1(t)$ and $x_2(t)$:

$$x_1(t) = r(t) coshu(t) \quad , \quad x_2(t) = r(t) sinhu(t) \quad , \tag{2.9}$$

in terms of which the oscillator coordinate $r(t)$ is indeed expressed in the pseudo-euclidean metrics:

$$r(t)^2 \equiv x_1(t)^2 - x_2(t)^2 \quad . \tag{2.10}$$

Use of the (canonical) transformation

$$x(t) = \frac{x_1(t) + x_2(t)}{\sqrt{2}} = \frac{1}{\sqrt{2}} r(t) e^{u(t)} \quad , y(t) = \frac{x_1(t) - x_2(t)}{\sqrt{2}} = \frac{1}{\sqrt{2}} r(t) e^{-u(t)} \quad , \tag{2.11a}$$

in eqs.(2.1) readily gives eq.(2.7), provided

$$u(t) \equiv -\frac{\gamma}{2m} t \quad , \tag{2.11b}$$

as required by the time independence of the coefficients in eqs.(2.1). Vice versa, the oscillator (2.7) is *decomposed* into two damped oscillators (2.1) when the pseudo-euclidean metrics (2.10) is adopted and the transformation (2.11a-b) is used. Similarly, use of eqs.(2.11) in the Lagrangian (2.2) gives the Lagrangian for the harmonic oscillator (2.7):

$$L = \frac{1}{2} m \dot{r}^2 - \frac{1}{2} K r^2 \tag{2.12}$$

(and vice versa (2.2) can be obtained from (2.12)).

Of course, if one chooses the euclidean metrics, $r^2 \equiv x_1^2 + x_2^2$, $x_1 = r cos\alpha$, $x_2 = r sin\alpha$, the r-oscillator is *decomposed* into two *undamped* oscillators.

These observations, although of elementary simplicity, teach us the rôle of the pseudo-euclidean metrics in setting up the canonical formalism for dissipative systems and clarify the meaning of the doubling of the degrees of freedom.

In the presence of the damping factor $e^{-\Gamma t}$ forward time evolution cannot be mapped (transformed) into backward time evolution by any operation (as complex conjugation in the non-dissipative case) except time-reversal $t \to -t$

(or, which is the same, by changing the relative sign of Γ and t; here we keep fixed the sign of Γ). In dissipative systems, however, time-reversal symmetry is broken, which means that one cannot use time-reversal $t \to -t$ without changing the physics; the conclusion is that dissipation induces a *partition* on the time axis thus implying that positive and negative time directions necessarily must be associated with *separate* modes, respectively describing different physical situations.

The canonical formalism, which only describes closed systems, needs therefore to be constructed on a set of *double* modes, one for each direction of time evolution (each other time-reversed), and these modes are indeed provided by hyperbolic coordinates (see eqs.(2.11)).

In the non-dissipative case, where only oscillating factors of type $e^{\pm iEt}$ are involved, one can actually limit himself to consider, e.g., only forward time direction, the backward time direction being obtained by complex conjugation operation (or by $t \to -t$ which is now allowed since time-reversal is not broken); this is why one does not need to consider backward and forward modes as *separate* modes in the non-dissipative systems.

We can conclude that the *decomposition* of a canonical (Hamiltonian) system into two subsystems needs pseudo-euclidean metrics when these subsystems are noncanonical ones. Vice versa, setting up the canonical formalism for a dissipative system requires (as well known) the inclusion into the formalism of the reservoir and this can be achieved by doubling the system (the degrees of freedom) which in turn implies the pseudo-eulidean metrics. We also remark that the *decomposition* operation can be shown to be formally expressed at the algebraic level by the co-product operation of Hopf algebras[15]. We will not discuss this point which is out of the task of the present paper. Details on the relation with coalgebra and quantum deformation of Lie groups may be found in refs.[16,17]. It should be finally stressed that doubling of degrees of freedom is a central ingredient not only in dissipation theory but also in thermal field theories[11,18] and in the C^*-algebraic formalism for statistical mechanics[19].

In terms of the hyperbolic coordinates x_1 and x_2 the Lagrangian (2.2) is

$$L = L_{0,1} - L_{0,2} + \frac{\gamma}{2}(\dot{x}_1 x_2 - \dot{x}_2 x_1) \quad , \tag{2.13}$$

with

$$L_{0,i} = \frac{m}{2}\dot{x}_i^2 - \frac{k}{2}x_i^2 \quad , i = 1, 2. \tag{2.14}$$

The associate momenta are

$$p_1 = m\dot{x}_1 + \frac{\gamma}{2}x_2 \quad , \quad p_2 = -m\dot{x}_2 - \frac{\gamma}{2}x_1 \tag{2.15}$$

and the motion equations corresponding to the set (2.1) are

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0 \quad , \tag{2.16a}$$

$$m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0 \quad . \tag{2.16b}$$

The Hamiltonian (2.4) becomes

$$H = H_1 - H_2 = \frac{1}{2m}(p_1 - \frac{\gamma}{2}x_2)^2 + \frac{k}{2}x_1^2 - \frac{1}{2m}(p_2 + \frac{\gamma}{2}x_1)^2 - \frac{k}{2}x_2^2 \quad . \tag{2.17}$$

It is interesting to write down (2.17) for the generic metrics $g^{ij} = g_{ij} = diag(1, \kappa^2)$, where $\kappa^2 = +1$ for euclidean spatial plane, and $\kappa^2 = -1$ for pseudo-euclidean plane:

$$H = \frac{1}{2m}(p_1 + \kappa^2\frac{\gamma}{2}x_2)^2 + \frac{k}{2}x_1^2 + \frac{1}{2m}\kappa^2(p_2 - \kappa^2\frac{\gamma}{2}x_1)^2 + \kappa^2\frac{k}{2}x_2^2 \quad . \tag{2.18}$$

Eq.(2.13) shows that the dissipative term actually acts as a coupling between the oscillators x_1 and x_2 on the hyperbolic plane. On the other hand, eq.(2.17) shows that the damping manifests as a correction in the kinetic energy for both oscillators. In refs.[7,12] the Hamiltonian of quantum dissipation is obtained from eq.(2.4) (or (2.17)) by using the canonical operators (2.5) and it is shown[7] to be the same as the one of Thermo Field Dynamics[11] for the boson case. In both cases, in dissipative dynamics as well as in Thermo Field Dynamics, the underlying group structure is the one of SU(1,1), the free Hamiltonian being the SU(1,1) Casimir operator[7]: $H_{0,1} - H_{0,2}$.

In eqs.(2.13) and (2.17) it is remarkable the occurrence of the minus sign between $L_{0,1}$ and $L_{0,2}$ and H_1 and H_2 , respectively, which is in fact derived as a direct consequence of the pseudo-euclidean metrics (cf.(2.18)) (such a minus sign is, on the contrary, postulated in Thermo Field Dynamics). In quantum dissipation and in Thermo Field Dynamics the degeneracy of the ground state is strictly related to this special form of the free Hamiltonian[7,11]. We observe that no problem arises with the boundness from below of H in (2.17) since it is constant of motion and once its positive value is given it is preserved in time.

We note that the generator of the canonical transformations (2.9) is[20]

$$\mathcal{U}(p_1,p_2;r,u) \equiv p_1 r coshu + p_2 r sinhu \quad , \tag{2.19}$$

such that $x_i = \frac{\partial \mathcal{U}}{\partial p_i}$, $i = 1,2$, and $p_r = \frac{\partial \mathcal{U}}{\partial r} = p_1 coshu + p_2 sinhu$, $p_u = \frac{\partial \mathcal{U}}{\partial u} = p_1 r sinhu + p_2 r coshu$. One then easily derives the invariant form

$$p_1{}^2 - p_2{}^2 = p_r{}^2 - \frac{1}{r^2} p_u{}^2 \quad , \tag{2.20}$$

where again we notice the appearance of the minus sign. By observing that $p_r = m\dot{r}$ and $p_u = 0$ and by adding to both members of (2.20) the invariant (under (2.9)) quantity $\frac{k}{2} r^2$ we recover again (2.17) and its equality with the Hamiltonian of the harmonic oscillator r: $\frac{1}{2} m\dot{r}^2 + \frac{1}{2} K r^2$.

By following ref.[13] we now introduce the vector potential

$$A_i = \frac{B}{2} \epsilon_{ij} x_j \quad , \quad i,j = 1,2 \ , \quad B \equiv \frac{c}{e} \gamma \quad , \tag{2.21}$$

with $\epsilon_{ii} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$, and the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = -B\hat{\boldsymbol{3}}$. Then H_i, $i = 1,2$, in eq.(2.17) describe two particles with opposite charge $e_1 = -e_2 = e$ in the (oscillator) potential $\Phi \equiv \frac{k}{2e}(x_1{}^2 - x_2{}^2) \equiv \Phi_1 - \Phi_2$, $\Phi_i \equiv \frac{k}{2e} x_i{}^2$ and in the constant magnetic field $\boldsymbol{B}$:

$$H = H_1 - H_2 = \frac{1}{2m}(p_1 - \frac{e_1}{c} A_1)^2 + e_1 \Phi_1 - \frac{1}{2m}(p_2 + \frac{e_2}{c} A_2)^2 + e_2 \Phi_2 \quad . \tag{2.22}$$

In fact eqs.(2.16) are nothing but the expressions of the Lorentz forces on particles with charge $e_1 = -e_2 = e$, in the electric field $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$, and in the magnetic field $\boldsymbol{B} = -B\hat{\boldsymbol{3}}$:

$$\mathcal{F}_i = m\ddot{x}_i = e_i [E_i + \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{B})_i] \quad , \ i = 1,2 \ , \tag{2.23}$$

with $\boldsymbol{v} = (\dot{x}_1, \dot{x}_2, 0)$. By using (2.21) the Lagrangian (2.13) is rewritten in the familiar form

$$\begin{aligned} L &= \frac{1}{2m}(m\dot{x}_1 + \frac{e_1}{c} A_1)^2 - \frac{1}{2m}(m\dot{x}_2 + \frac{e_2}{c} A_2)^2 - \frac{e^2}{2mc^2}(A_1{}^2 + A_2{}^2) - e\Phi \\ &= \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{e}{c}(\dot{x}_1 A_1 + \dot{x}_2 A_2) - e\Phi \quad , \end{aligned} \tag{2.24}$$

from which eqs.(2.23) are derived in the form

$$\frac{d}{dt}(m\dot{x}_i + \frac{e_i}{c}A_i) = -e_i\partial_i\Phi_i + \frac{e_i}{c}\partial_i v_j A_j \quad , \; i \neq j \; , no \quad sum \quad on \quad i, j \; , \qquad (2.25)$$

where ∂_i denotes $\frac{\partial}{\partial x_i}$. Note that $\frac{d}{dt}A_i = v_j\partial_j A_i$.

In the following sections we consider the relation with Chern-Simons gauge term and we study the dynamics of Bloch electrons in solids as a physical application.

3. Topologically massive Chern-Simons gauge theory in pseudo-euclidean space

We consider 2+1 dimensional space-time with a metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = diag(1, -1, -\kappa^2) \quad , \qquad (3.1)$$

where $\kappa^2 = +1$ for euclidean space plane, and $\kappa^2 = -1$ for pseudo- euclidean plane. Then the Abelian gauge field theory is described by the Maxwell form

$$L_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\{(F_{01}^2 + \kappa^2 F_{02}^2) - \kappa^2 F_{12}^2\} \quad , \qquad (3.2)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \qquad (3.3)$$

For the description of the infrared limit[6] usually a gauge non-invariant mass term can be introduced in the Proca form[21]

$$L_P = \frac{1}{2}\mu^2 A_\mu A^\mu = \frac{1}{2}\mu^2(A_0^2 - A_1^2 - \kappa^2 A_2^2) \quad . \qquad (3.4)$$

On the other hand, the Chern-Simons term with statistical parameter θ acts as a gauge invariant topological mass term[1-5]

$$L_{CS} = \frac{\theta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \quad . \qquad (3.5)$$

We note that the Chern-Simons term (3.5) does not depend on the (euclidean or pseudo-euclidean) metrics one chooses.

General, three-dimensional topological massive Maxwell-Chern-Simons-Proca electrodynamics in the Weyl gauge, $A_0 = 0$, has the Lagrangian density

$$L = L_M + L_{CS} + L_P = \frac{1}{2}\{(\dot{A}_1^2 + \kappa^2 \dot{A}_2^2) - \kappa^2(\boldsymbol{\nabla} \times \boldsymbol{A})^2\}$$

$$+\frac{\theta}{2}|\dot{\boldsymbol{A}} \times \boldsymbol{A}| - \frac{1}{2}\mu^2(A_1^2 + \kappa^2 A_2^2) \quad . \tag{3.6}$$

We now remark that, despite its independence of the metrics, the *topological* term leads to drastically different situations depending if one works in the euclidean , $\kappa^2 = +1$, or in pseudo-euclidean , $\kappa^2 = -1$, space metrics.

In the euclidean case we have topologically massive electrodynamics[1-5]. For very massive modes when $\theta \to \infty$ the leading contribution is from the Chern-Simons term. It is well known that (euclidean) Chern-Simons gauge theory with matter field describes particles with fractional statistics - anyons[1-5]. In this case the topological mass parameter θ plays the rôle of statistical parameter and defines the spin of the particles.

Our statement is now that in the pseudo-euclidean case the Chern-Simons term generates dissipation in the infrared limit. Indeed, infrared properties of the theory are related to the constant in space vector potential $q_i(t)$:

$$A_i(x,t) = q_i(t) + \tilde{A}_i(x,t) \quad , \quad for\ \boldsymbol{x} \to \infty \ \ with\ \tilde{A}_i(x,t) \to 0 \quad , \tag{3.7}$$

and the corresponding Lagrangian density (3.6) becomes in the infrared limit

$$L = \frac{1}{2}(\dot{q}_1^2 + \kappa^2 \dot{q}_2^2) + \frac{\theta}{2}|\dot{\boldsymbol{q}} \times \boldsymbol{q}| - \frac{1}{2}\mu^2(q_1^2 + \kappa^2 q_2^2) \quad . \tag{3.8}$$

For $\kappa^2 = -1$ this L has exactly the form (2.13) of the Lagrangian density for the damped harmonic oscillator, provided the Proca mass is identified with the oscillating frequency: $\mu^2 \equiv \omega^2 = k/m$, and the topological (Chern-Simons) mass θ is related to the damping parameter γ:

$$\theta = \frac{\gamma}{m} \quad . \tag{3.9}$$

We thus conclude that damping has a global, topological origin and cannot be obtained by usual local mass.

In this context interesting problems arise if we consider Chern-Simons gauge field interacting with matter fields in the pseudo-euclidean plane. The

dynamics of charged particle moving in the hyperbolic plane is studied in connection with problems of chaotic dynamics[22,23]. A connection exists between the classical behaviour of a system and its energy level fluctuations. Typical problems are the ones of the billiard in magnetic field with time-reversal symmetry breaking[23]. In the presence of time-reversal breaking and non-trivial flux (Aharonov-Bohm potential) the spectrum is entirely controlled by topology. When the classical trajectories are chaotic, such time-reversal symmetry breaking has a dramatic effect on the spectrum: it changes the universality class of the local statistics of high-lying energy levels[23]. Switching on of the flux leads to switching off the time-reversal invariance without changing the geometry of classical trajectories. If the chaotic motion of the system, like K-systems[24], with exponential separation of trajectories with time can be simulated by motion in the hyperbolic plane, and the Aharonov-Bohm statistical flux can be introduced by Chern-Simons term, the problem is formally very close to the framework above discussed. Indeed, the hyperbolic plane, as a simplest surface of constant negative curvature, can be globally embedded in a space endowed with a Minkowskian metrics instead of a Euclidean one (for surfaces of negative curvature such a global embedding in Euclidean space is in fact prohibited [26]. On the other hand only negative curvature leads to chaotic motion[27]). To be more specific on this point we study in the following section the case of hyperbolic energy surface in the dynamics of electrons in solids propagating along the plane of the lattice.

4. Dynamics of Bloch Electrons

As an application of physical interest we consider in this section the dynamics of Bloch electrons in solid. As well known[14], in such a problem a central rôle is played by the inverse effective mass tensor

$$\frac{1}{m} \to \frac{1}{\hbar^2}\frac{\partial^2}{\partial \boldsymbol{k}^2}E(\boldsymbol{k}) \quad . \tag{4.1}$$

In semiconductors the energy surface $E(\boldsymbol{k})$ has the form

$$E(\boldsymbol{k}) = \hbar^2(\frac{k_1^2}{2m_1} + \frac{k_2^2}{2m_2} + \frac{k_3^2}{2m_3}) \tag{4.2}$$

in a convenient coordinate system. The tensor (4.1) is not necessarily positive definite. When the Fermi surface is near the band boundary, the sign of the mass m_3 is negative[14]. In this *hyperbolic* case, due to strong Bragg reflection from the boundary of the band, the electron propagates along trajectories which are parallel to the planes of the lattice[14] and $E(\boldsymbol{k})$ has "saddle points", where, i.e., the curvature of the surface may be positive in one direction and negative in another one. In Ziman's words[14], "the dynamics of states in this region may be most unexpected". Indeed, as we show below the dynamics in this case has "dissipative" character.

We consider only 2-dimensional motion, which is the case when one of the mass, for example m_2, goes to infinity. We also restrict ourselves with the simple case $m_1 = |m_3|$. Thus, redefining $m_3 \rightarrow m_2$, we have the energy surface

$$E(\boldsymbol{k}) = \hbar^2 (\frac{k_1^2}{2m} + \kappa^2 \frac{k_2^2}{2m}) \quad , \tag{4.3}$$

and the effective mass matrix

$$M^{-1} = diag(\frac{1}{m}, \frac{\kappa^2}{m}) \quad . \tag{4.4}$$

Since Bloch electrons are subject to the Lorentz force[14], we have the Lagrangian for a particle in external magnetic field $B_3 = \epsilon_{3ij} \partial_i A_j$ and electric field $-\partial_i \Phi$:

$$L = \frac{m}{2}(\dot{x}_1^2 + \kappa^2 \dot{x}_2^2) + \frac{e}{c}(\dot{x}_1 A_1(\boldsymbol{x}) + \dot{x}_2 A_2(\boldsymbol{x})) - e\Phi(\boldsymbol{x}) \quad . \tag{4.5}$$

For rotationally symmetric motion in 2 dimensions $A_i(\boldsymbol{x})$ and $\Phi(\boldsymbol{x})$ are given by

$$A_i(\boldsymbol{x}) = \epsilon_{ij} x_j A(x) \quad , \tag{4.6}$$

$$\Phi(\boldsymbol{x}) = \Phi(x) \quad . \tag{4.7}$$

For simplicity we consider a constant magnetic field $A(x) = \frac{B}{2}$ and a quadratic scalar potential

$$\Phi(x) = \frac{k}{2e}(x_1^2 + \kappa^2 x_2^2) \quad . \tag{4.8}$$

Then from (4.5)

$$L = \frac{m}{2}(\dot{x}_1^2 + \kappa^2 \dot{x}_2^2) + \frac{e}{2c} B |\dot{\boldsymbol{x}} \times \boldsymbol{x}| - \frac{k}{2}(x_1^2 + \kappa^2 x_2^2) \tag{4.9}$$

and we have the same particle problem as in Sec.3, eq.(3.8). For $\kappa^2 = -1$ it is defined by the Lagrangian (2.13) (or equivalently (2.24)) and the magnetic field B plays role of the damping γ ($B \equiv \frac{c}{e}\gamma$).

5. Canonical structure and Chern-Simons limit

In previous sections we have obtained the Lagrangian (2.13) (or (2.24)) from the Chern-Simons gauge theory in the infrared limit (Sec.3) and from the Bloch electron dynamics (Sec.4). Therefore we can restrict to (2.13) the discussion of the canonical structure.

The canonical momenta are defined by eqs.(2.15) and the Hamiltonian is given by (2.17), while the canonical brackets are

$$\{x_i, p_j\} = \delta_{ij} \quad , \tag{5.1}$$

$$\{x_i, x_j\} = \{p_i, p_j\} = 0 \quad . \tag{5.2}$$

To extract the effects of the pure damping we consider the limit of strong damping, $\gamma >> m$. In this case we can neglect the kinetic term in the Lagrangian (2.13) and we have

$$L = +\frac{\gamma}{2}(\dot{x}_1 x_2 - x_1 \dot{x}_2) - \frac{k}{2}(x_1^2 - x_2^2) \quad . \tag{5.3}$$

The Euler-Lagrange equations now are

$$\begin{aligned} \gamma \dot{x}_1 + k x_2 = 0 \quad , \\ \gamma \dot{x}_2 + k x_1 = 0 \quad , \end{aligned} \tag{5.4}$$

with general solution

$$\begin{aligned} x_1(t) = a e^{\lambda t} + b e^{-\lambda t} \quad , \\ x_2(t) = -a e^{\lambda t} + b e^{-\lambda t} \quad , \end{aligned} \tag{5.5}$$

where $\lambda = \frac{k}{\gamma}$. This solution describes the damping of the coordinate $x = \frac{1}{\sqrt{2}}(x_1 + x_2)$ and the growing of the coordinate $y = \frac{1}{\sqrt{2}}(x_1 - x_2)$. It is worth to note that the damping coefficient $\lambda = \frac{k}{\gamma}$ in the limiting case is different from the original value $\Gamma = \frac{\gamma}{2m}$.

Another way to proceed simply consists in putting the mass $m = 0$ in eq.(2.13) (and (2.16)). We need however to comment upon some peculiarity

of this reduction procedure [25]. Under the reduction, 4-dimensional phase space for the second-order system (2.16) turns into 2-dimensional one $\{x_1, x_2\}$ for the first-order system (5.4). The first-order system is called the "degenerate system"[25]: An arbitrary initial value problem for (2.16), in general, does not apply to the degenerate system (5.4). For eqs.(2.16), at $t = 0$ we can attach an arbitrary value for coordinates x_1, x_2 and related velocities $\dot{x}_1, \dot{x}_2$. However, only after some time interval we can describe the same physical system by eqs.(5.4) and, moreover, in such a case $\dot{x}_1$ and $\dot{x}_2$ cannot be arbitrary since they are completely determined by given coordinates x_1, x_2, according to (5.4)(see eq.(5.8)). Subsequent evolution of the system is determined by the potential (5.6) only, and depends on the signature of the corresponding bilinear form. When the mass m goes to zero, transition from a state incompatible with eq.(5.4) to a compatible one is very fast. Acceleration at the initial time is very high (related velocity is changing very fast). The transition to the massless limit can be well approximated by the discontinuous jumping condition [25]: the energy of the system cannot be changed by jump. In our case, all energy is concentrated in the potential (5.6) (the kinetic energy is vanishing in the massless limit $m \to 0$). The jumping condition thus implies that under the jumping the coordinates of the system remain invariant and only the velocities can be changed.

The limiting case has analogs in the theories described above in Secs.3 and 4. In terms of the topologically massive Maxwell-Chern-Simons-Proca theory (3.6) it corresponds to the so called Chern-Simons limit, valid for long wave excitations. Since pure Chern-Simons theory is meaningful only for topologicaly nontrivial 3-manifolds and the Chern-Simons term does not contribute to the energy-momentum tensor and has vanishing Hamiltonian, to have local, propagating modes we must keep the Proca mass term (3.4).

For the Bloch electron model (4.5) the strong damping limit corresponds to strong external magnetic and electric fields, when we can neglect the electron effective mass m.

Since the reduction procedure for Chern-Simons theories has already been discussed in the literature[2], we only discuss the reduction in the context of the damping problem.

The reduced theory is described by the first-order Lagrangian (5.3). The Hamiltonian is simply

$$H = \frac{k}{2}(x_1^2 - x_2^2) \tag{5.6}$$

and, as shown below, the symplectic structure of the reduced model (5.3) is

$$\{x_i, x_j\} = \frac{1}{\gamma}\epsilon_{ij} \quad . \tag{5.7}$$

The Hamilton equations of motion are

$$\dot{x}_1 = \{H, x_1\} = -\frac{k}{\gamma}x_2 \quad , \quad \dot{x}_2 = \{H, x_2\} = -\frac{k}{\gamma}x_1 \quad , \tag{5.8}$$

coinciding with the Euler-Lagrange equations (5.4).

The bracket (5.7) follows from the canonical one (5.1) by Dirac procedure for constrained systems. Indeed, in the limit $m \to 0$ from eqs.(2.15) for the canonical momenta we have the two following constraints

$$C_1 = p_1 - \frac{\gamma}{2}x_2 \quad , \quad C_2 = p_2 + \frac{\gamma}{2}x_1 \quad , \tag{5.9}$$

which are second class as

$$G_{ij} = \{C_i, C_j\} = -\gamma\epsilon_{ij} \quad . \tag{5.10}$$

Using the definition of Dirac brackets :

$$\{A, B\}_D = \{A, B\}_P - \{A, C_i\}(G^{-1})_{ij}\{C_j, B\} \quad , \tag{5.11}$$

where

$$(G^{-1})_{ij} = \frac{1}{\gamma}\epsilon_{ij} \quad , \tag{5.12}$$

we obtain the following Dirac brackets

$$\{x_i, x_j\}_D = \frac{1}{\gamma}\epsilon_{ij} \quad , \tag{5.13a}$$

$$\{p_i, p_j\}_D = \frac{\gamma}{4}\epsilon_{ij} \quad , \tag{5.13b}$$

$$\{x_i, p_j\}_D = \frac{1}{2}\delta_{ij} \quad . \tag{5.13c}$$

From the first brackets we recognize eq.(5.7) as the Dirac bracket for constrained damped oscillator model (2.13).

Let us conclude this section with few remarks.

We observe that the pseudo-euclidean character of the kinetic term in the original model (2.13) does not affect the constrained theory. The essential difference between the euclidean case and the pseudo-euclidean one relies in the sign of the second term in the Hamiltonian (5.6), i.e. in the signature of the quadratic form. The bracket (5.7) is the same in both the metrics and is independent of the structure (5.6). However, the physics is drastically different in the two cases: in the euclidean metrics we have oscillating motion, while in the pseudo-euclidean metrics we have damping phenomena. The global, topological nature of the damping phenomena is shown by the non-analytic dependence on the damping γ of the bracket (5.7) (or of the Hamilton equations (5.8)).

The above remarks suggest few more observations on Chern-Simons gauge field theory and on the electron motion in external field.

For the gauge theory in the pure Chern-Simons limit, an addition of anisotropic Proca mass term can lead to unusual phenomena for the light behaviour.

For the electron motion in a strong magnetic and electric fields, the dynamics is independent of the electron effective mass and is completely determined by the structure of the electric field. For the harmonic electric potential with positive signature (the euclidean case), the dynamics has an oscillating character. While for a "saddle point" potential the harmonic force with opposite coupling in different directions leads to damping mode in one direction and growing mode in the another one. Actually, the solution (5.5) describes the motion on the hyperboloid

$$x_1^2(t) - x_2^2(t) = 4ab = const \quad . \tag{5.14}$$

6. Conclusions

In this paper we have shown that in the pseudo-euclidean metrics the Lagrangian density of 2+1 dimensional gauge theory with Chern-Simons term and Proca mass term has, in the infrared region, the same form of the Lagrangian density of the damped harmonic oscillator. In the pseudo-euclidean metrics Chern-Simons gauge theory in the infrared region can be thus associated with dissipative dynamics. As an explicit example of practical interest, we have considered the propagation of Bloch electrons in solids along the plane of the lattice with hyperbolic energy surface.

In order to exhibit the rôle played by pseudo-euclidean metrics in the canonical formalism for dissipation we have shown that the system of a damped harmonic oscillator and of its time-reversed image, this last one introduced by doubling the degrees of freedom as required indeed by canonical formalism, actually behaves as a closed system described by a single harmonic oscillator. More generally, it is also true that by use of hyperbolic coordinates a closed system may be "decomposed" into two open subsystems, each one acting as the reservoir for the other one.

The equations for the damped oscillator and for its time-reversed image have been proven to be the same as the ones for the Lorentz force acting on two particles carrying opposite charge $e_1 = -e_2 = e$ in a constant magnetic field (simulating the damping factor) and in the electric harmonic potential. Such a representation provides an immediate link with the above mentioned example of Bloch electrons in solids and with the dynamics of charged particle in the hyperbolic plane, which is studied also in connection with problems of chaotic motion[23].

Our analysis has been carried on at the classical level. However, the canonical quantization of the damped harmonic oscillator has been worked out in the operator formalism in ref.[7] and its relation with Feynmann-Vernon functional integral scheme and with Schwinger formalism for dissipative systems has been analysed in ref.[8]. As shown in [7] the set of states of the system splits into unitarily inequivalent representations of the canonical commutation relations and the irreversibility of time evolution is expressed in terms of tunneling among the unitarily inequivalent representations. The underlying group structure of the theory is the one of $SU(1,1)$ and the vacuum state is an $SU(1,1)$ time dependent coherent state. In view of the results presented in this paper we expect that the canonical quantization scheme of refs.[7-9] can be extended to the Maxwell-Chern-Simons-Proca gauge theory in the infrared region in pseudo-euclidean metrics. We thus expect that also in the case of Chern-Simons theories in pseudo-euclidean metrics a proper quantization scheme requires the full set of unitarily inequivalent representations of the canonical commutation rules. In the same way, the thermal nature of the vacuum state in the damped oscillator case[7] and its relation with thermal vacuum state of quantum field theory at finite temperature in real time (Thermo Field Dynamics[11]) may lead to interesting features in extending the canonical quantization to pseudo-euclidean Chern-Simons gauge theories.

Finally, we observe that the time evolution generator of damped oscillator can be expressed in terms of quantum deformation of the Weyl-Heisenberg algebra[16] and that the coproduct operation of quantum deformed Lie-Hopf algebras play a relevant rôle in thermal field theories[17]. From the discussion of this paper it is then to be expected that quantum deformations of Lie-Hopf algebras also play a relevant rôle in pseudo-euclidean Chern-Simons theories. On this subject we plan to present our analysis in a future work.

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